An asymptotic analysis of the so called intelligent PID controller

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This note takes its origin from reading a paper by M. Fliess and C. Join, whose latest version is entitled *Model-free control and intelligent PID controllers: towards a possible trivialization of nonlinear control?* that can be obtained at `http://arxiv.org/abs/0904.0322`

Unfortunately in that paper, the authors write that *... it is impossible of course to give a complete description* of it and that *the usual mathematical criteria for robust control become ... irrelevant.*

In the following, to obtain an usable description of a controller and to give some guarantees that it performs correctly, we interpret and materialize the ideas proposed in that paper. Of course, this is at the price of trivializing and likely also degrading them. Nevertheless, this allows us to present an elementary analysis using standard mathematical criteria although we know the authors claimed that this is irrelevant.

1 Controller design

To describe the controller, we start by assuming, but just as a design tool, that the dynamics of the system to be controlled can be described simply by a disturbed integrator :

$$\dot{y} \;=\; b\,u \;+\; d$$

where y is the system output, u is the control and d is the disturbance. For such an elementary model, a compensating output feedback is trivially :

$$u \;=\; \frac{-\widehat{d} + v}{b} \tag{1}$$

where $\widehat{d}$ is an estimation of the disturbance d and v is the control of the compensated system. To implement the above feedback we need a disturbance estimator. There is a vast literature devoted to disturbance estimation. We solve this problem below using an estimation of $\dot{y}$.

For a given signal y, its derivative at time $t - T$ can be estimated as[1] :

$$\widehat{\dot{y}}(t-T) \;=\; \frac{6}{T}\int_0^1 (1-2\sigma)y(t-\sigma T)d\sigma \;,$$

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[1]The motivation for this expression is coming from standard interpolation theory.

provided T is small enough. Indeed the transfer function F_1 of the operator on the right hand side is :

$$\begin{aligned} F_1(s) &= \frac{6 \displaystyle\int_0^1 (1-2\sigma)\ \exp(-s\sigma T) d\sigma}{T} \\ &= 6 \frac{-2 + sT + (sT+2)\exp(-sT)}{s^2 T^3} \\ &= s \ -\ \frac{1}{2} s^2 T \ +\ \frac{3}{20} s^3 T^2 \ +\ \frac{o(s^3T^3)}{T}\ . \end{aligned}$$

Now for d satisfying :

$$d \ = \ \dot{y} - bu$$

we can propose an estimate of its value at time $t-T$ as :

$$\widehat{d}(t-T) \ = \ \frac{6}{T} \int_0^1 (1-2\sigma) y(t-\sigma T) d\sigma \ - \ b\, u(t-T)\ .$$

With this, the compensating feedback (1) takes the form :

$$u(t) \ = \ -\frac{\frac{6}{T} \displaystyle\int_0^1 (1-2\sigma) y(t-\sigma T) d\sigma \ - \ v}{b} \ + \ u(t-T)\ .$$

For instance, choosing a simple proportional controller to track a reference signal y_r, we get the controller :

$$u(t) \ = \ -\frac{\frac{6}{T} \displaystyle\int_0^1 (1-2\sigma) y(t-\sigma T) d\sigma \ - \ [\dot{y}_r(t) - k(y(t) - y_r(t))]}{b} \ + \ u(t-T) \qquad (2)$$

Its transfer function is :

$$C_{1P}(s) \ = \ -\frac{\dfrac{6[-2 + sT + (sT+2)\exp(-sT)]}{s^2T^3} + k}{b[1 - \exp(-sT)]}$$

The parameters of this controller are b, T and k.

Note that, since we have no model for the system, it is tricky to plan meaningful paths y_r for the output. As a consequence, convergence to zero of the tracking error is in general impossible[2]. Also, when T is very small, C_{1P} can be approximated by $-\left(\frac{1}{bT} + \frac{k}{bT}\frac{1}{s}\right)$. This corresponds to an elementary PI controller. So :

The controller (2) is nothing but a complicated approximation of an elementary PI controller.

For these reasons we call the controller (2) the half-wit's P controller.

[2]For a finite dimensional linear system with $\frac{B(s)}{A(s)}$ as transfer function, exact tracking is possible only if y_r is in the kernel of $\ b[1 - \exp(-sT)]\, A(s) \ + \ [F_1(s) - s]\, B(s)$.

2 Analysis of a closed loop system

We study the case where the half-wit's P controller is applied to a system with transfer function $\frac{B(s)}{A(s)}$ with relative degree 1 and order n. The poles of the closed loop system are the zeros of :

$$R(z) \;=\; b[1-\exp(-zT)]\,A(z) \;+\; \left(6\frac{-2+zT+(zT+2)\exp(-zT)}{z^2T^3}+k\right)B(z)\;.$$

When T is very small, for all complex variable z so that $|z| \ll \frac{1}{T}$, $R(z)$ can be approximated by the following polynomial Q of degree $n+1$:

$$R(z) \;\approx\; Q(z) \;=\; bzT\,A(z) \;+\; (z+k)\,B(z)$$

With b and k fixed, when T tends to 0, $n-1$ (the degree of B) roots of Q tend to the roots of B, one root tends to $-k$ and the last root tends to $-\frac{\beta_{hf}}{bT}$, where β_{hf} is the coefficient of z^{n-1} in $B(z)$. So with the help of Rouche's Theorem, if $\frac{|\beta_{hf}|}{|b|}$ is small enough with respect to 1, if the roots of B have strictly negative real part, and b is chosen with same sign as β_{hf}, the zeros of R should all have negative real part. We have "established" :

By picking T small enough, the half-wit's P controller is able to stabilize any finite dimensional minimum phase linear system with relative degree 1 for which we know the sign of the high frequency gain.

With k fixed, T tending to 0, but bT going to infinity, the n roots of Q go to the roots of A and the last root tends to $-\frac{k\beta_s}{bT\alpha_s}$, where α_s and β_s are the constant coefficients of A and B respectively, assuming $\alpha_s \neq 0$. So, if the roots of A have strictly negative real part, and kb is chosen with the same sign as the static gain of the system, the zeros of R should all have negative real part. We have "established" :

By picking T small enough and bT large enough, the half-wit's P controller is able to stabilize any finite dimensional stable linear system with relative degree 1 for which we know the sign of the static gain.

The above two conclusions are very well known and can be found in any text book dealing with PID controllers and root locus. There extensions in the multivariable case is also well known.

3 Extension

For the control v of the compensated system, we can replace the proportional action by a proportional + integral action. This leads to the controller transfer function :

$$C_{1PI}(s) \;=\; -\frac{\dfrac{6[-2+sT+(sT+2)\exp(-sT)]}{sT^3}+[k_Ps+k_I]}{bs[1-\exp(-sT)]}\;.$$

When T is very small, C_{1PI} can be approximated by $-\left(\frac{1}{bT}+\frac{k_P}{bT}\frac{1}{s}+\frac{k_I}{bT}\frac{1}{s^2}\right)$ which corresponds to a standard PI^2 controller.

We could even more generally think of a proportional + integral + derivative controller with transfer function :

$$C_{1PID}(s) \;=\; -\frac{(1+\bar{k}_D)\dfrac{6[-2+sT+(sT+2)\exp(-sT)]}{sT^3}+[\bar{k}_Ps+\bar{k}_I]}{\bar{b}s[1-\exp(-sT)]}$$

where we have used our estimate of $\dot{y}$ for the D component. But this is a fake extension since we recover C_{1PI} by changing the parameters as follows :

$$b = \frac{\bar{b}}{1+\bar{k}_D} \quad , \qquad k_P = \frac{\bar{k}_P}{1+\bar{k}_D} \quad , \qquad k_I = \frac{\bar{k}_I}{1+\bar{k}_D} \ .$$

Finally instead of using a disturbed integrator as model, we can use a double integrator i.e.

$$\ddot{y} = b\,u + d$$

Then to estimate $\ddot{y}$, we can use :

$$\widehat{\ddot{y}}(t-T) = \frac{60}{T^2}\int_0^1 (1-6\sigma+6\sigma^2)y(t-\sigma T)d\sigma$$

associated to the transfer function :

$$\begin{aligned} F_2(s) &= \frac{60\int_0^1 (1-6\sigma+6\sigma^2)\,\exp(-s\sigma T)d\sigma}{T^2} \\ &= 60\frac{(s^2T^2-6sT+12)-(s^2T^2+6sT+12)\exp(-sT)}{s^3T^5} \\ &= s^2 - \frac{1}{2}s^3T + \frac{o(s^3T^3)}{T^2} \ . \end{aligned}$$

Hence with a PID controller for the control v of the compensated system, we get the following transfer function for the feedback :

$$C_{2PID}(s) = -\frac{sF_2(s)+k_DsF_1(s)+k_Ps+k_I}{bs(1-\exp(-sT))}$$

which, when T is very small, can be approximated by a PI^2D controller.

And of course we can go on with a model coming from a chain of p integrators. This leads to a controller whose transfer function can be approximated, when T is very small, by the following very particular rational function :

$$C_{pPID}(s) \approx \frac{\displaystyle\sum_{i=0}^{p+1} c_i s^i}{bTs^2(\varepsilon s+1)^{p-1}}$$

where we have introduced the extra term $(\varepsilon s+1)^{p-1}$ just to guarantee properness[3]. Again invoking Rouche's Theorem and root locus, it can be checked that the parameters c_i and ε can be tuned[4] so that this controller can stabilize any finite dimensional minimum phase linear system with relative degree p provided we know an upperbound and the sign of its high frequency gain.

[3]A standard way to obtain a causal transfer function for the controller is to use the Pade approximation of the exponential function. For instance we can pick the (p,p)-approximation

$$exp(-sT) = \frac{\sum_{i=0}^{p}(-1)^i\frac{p!(2p-i)!}{(p-i)!(2p)!i!}(sT)^i}{\sum_{i=0}^{p}\frac{p!(2p-i)!}{(p-i)!(2p)!i!}(sT)^i}$$

which corresponds to a stable finite dimensional linear system.

[4]Pick the numerator of the controller transfer function as $\frac{\delta}{\varepsilon}$ times a Hurwitz polynomial, with δ chosen with the same sign as the high frequency gain and smaller than a threshold depending only on p and the upperbound for the high frequency gain and with ε sufficiently small.

4 Conclusion

We have "established" that the half-wit's P controller and its many extensions do not bring anything new compared to finite dimensional linear controllers when the interval for evaluating the derivatives is chosen with a very small length.